

3

Polynomials



Let's study.

- Introduction to Polynomials
- Degree of the polynomial
- Value of the polynomial
- Operations on polynomials
- Synthetic Division
- Remainder theorem



Let's discuss.

$p^3 - \frac{1}{2}p^2 + p$; $m^2 + 2n^3 - \sqrt{3}m^5$; 6 are all algebraic expressions.

Teacher : Dear Students, consider each term of the expressions $p^3 - \frac{1}{2}p^2 + p$, $m^2 + 2n^3 - \sqrt{3}m^5$, 6 and state the power of each variable.

Madhuri : In the expressions $p^3 - \frac{1}{2}p^2 + p$ powers of p are 3, 2, 1 respectively.

Vivek : Sir, in the expression $m^2 + 2n^3 - \sqrt{3}m^5$ the powers of the variable are 2, 3, 5 respectively.

Rahul : Sir, apparently there is no variable in the expression 6. But $6 = 6 \times 1 = 6 \times x^0$. Therefore, the power of the variable is 0.

Teacher : In all algebraic expressions given above the powers of the variable are positive integers or zero. i.e. whole numbers.

In an algebraic expression, if the powers of the variables are whole numbers then that algebraic expression is known as **polynomial**. 6 is also a polynomial. 6, -7, $\frac{1}{2}$, 0, $\sqrt{3}$ etc. are constant numbers can be called as **Constant polynomial**. 0 is also a constant polynomial.

Are $\sqrt{y} + 5$ and $\frac{1}{y} - 3$ polynomials?

Sara : Sir, $\sqrt{y} + 5$ is not a polynomial, because $\sqrt{y} + 5 = y^{\frac{1}{2}} + 5$, here power of y is $\frac{1}{2}$ which is not a whole number.

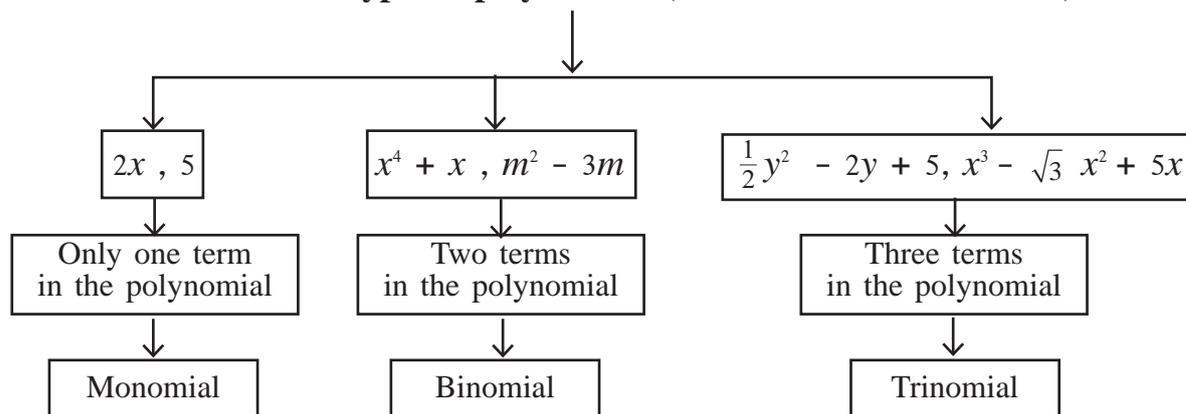
John : Sir, $\frac{1}{y} - 3$ is also not a polynomial because $\frac{1}{y} - 3 = y^{-1} - 3$, here power of y is -1 which is not a whole number.

Teacher : Write any five algebraic expressions which are not polynomials.

Explain why these expressions are not polynomials ? Justify your answer.

- Is every algebraic expression a polynomial ?
- Is every polynomial an algebraic expression ?

Types of polynomials (based on number of terms)



Polynomials are written as $p(x)$, $q(m)$, $r(y)$ according to the variable used.

For example, $p(x) = x^3 + 2x^2 + 5x - 3$, $q(m) = m^2 + \frac{1}{2}m - 7$, $r(y) = y^2 + 5$



Let's learn.

Degree of a polynomial in one variable

Teacher : In the polynomial $2x^7 - 5x + 9$ which is the highest power of the variable ?

Jija : Sir, the highest power is 7.

Teacher : In case of a polynomial in one variable, the highest power of the variable is called the **Degree of the polynomial**.

Now tell me, what is the degree of the given polynomial?

Ashok : Sir, the degree of the given polynomial $2x^7 - 5x + 9$ is 7.

Teacher : What is the degree of the polynomial 10 ?

Radha : $10 = 10 \times 1 = 10 \times x^0$ therefore the degree of the polynomial 10 is 0.

Teacher : Just like 10, **degree of any non zero constant polynomial is 0. Degree of zero polynomial is not defined.**

Degree of a polynomial in more than one variable

The highest sum of the powers of variables in each term of the polynomial is the degree of the polynomial.

Ex. $3m^3n^6 + 7m^2n^3 - mn$ is a polynomial in two variables m and n . Degree of the polynomial is 9. (as sum of the powers $3 + 6 = 9$, $2 + 3 = 5$, $1 + 1 = 2$)

Activity I : Write an example of a monomial, a binomial and a trinomial having variable x and degree 5.

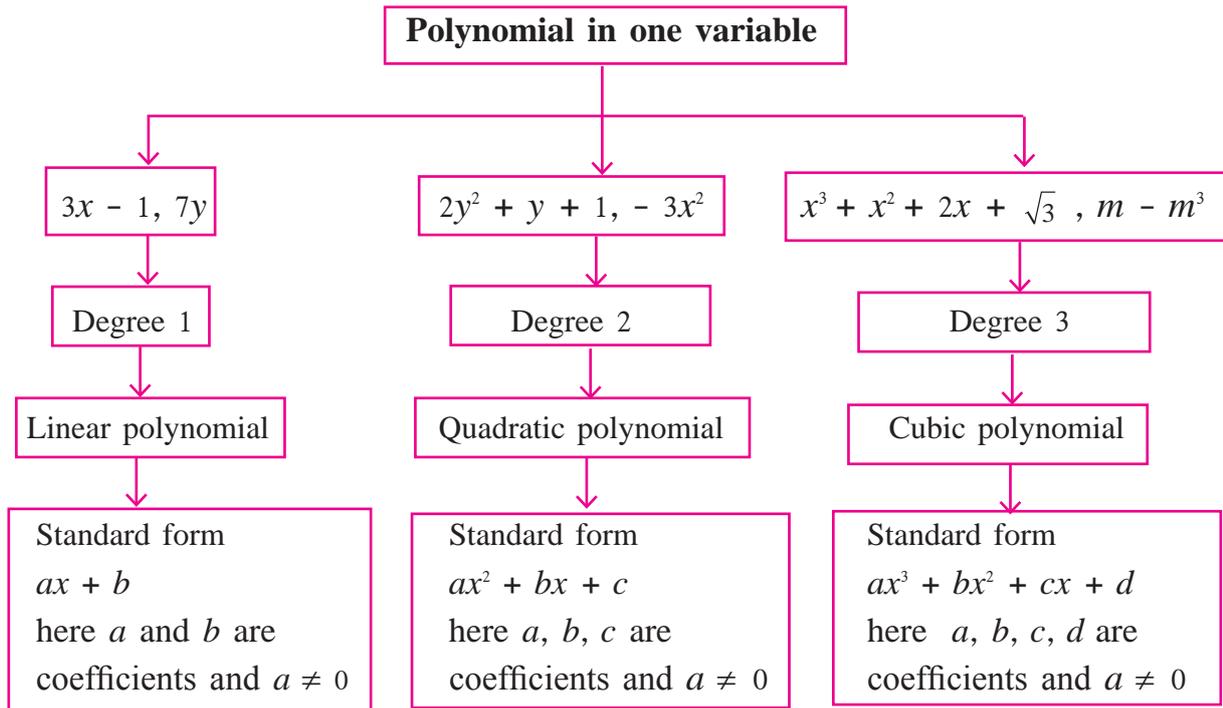
Monomial

Binomial

Trinomial

Activity II : Give example of a binomial in two variables having degree 5.

Types of polynomial (based on degree)



Polynomial : $a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0$ is a polynomial in x with degree n

$a_n, a_{n-1}, \dots, a_2, a_1, a_0$ are the coefficients and $a_n \neq 0$

Standard form, coefficient form and index form of a polynomial

$p(x) = x - 3x^2 + 5 + x^4$ is a polynomial in x , which can be written in descending powers of its variable as $x^4 - 3x^2 + x + 5$. This is called the standard form of the polynomial. But in this polynomial there is no term having power 3 of the variable we can write it as $0x^3$. It can be added to the polynomial and it can be rewritten as $x^4 + 0x^3 - 3x^2 + x + 5$. This form of the polynomial is called **Index form** of the polynomial.

One can write the coefficients of the variables by considering all the missing terms in the standard form of the polynomial. For example : $x^3 - 3x^2 + 0x - 8$ can be written as $(1, -3, 0, -8)$. This form of the polynomial is called **Coefficient form**.

Polynomial $(4, 0, -5, 0, 1)$ can be written by using variable y as

$4y^4 + 0y^3 - 5y^2 + 0y + 1$. This form is called **Index form** of the polynomial.

Ex. $p(m) = 3m^5 - 7m + 5m^3 + 2$

Write the polynomial in standard form	$3m^5 + 5m^3 - 7m + 2$
Write it in the index form by considering all the missing terms with coefficient zero.	$3m^5 + 0m^4 + 5m^3 + 0m^2 - 7m + 2$
Write it in a coefficient form	$(3, 0, 5, 0, -7, 2)$
Degree of the polynomial	5

Ex (1) Write the polynomial $x^3 + 3x - 5$ in coefficient form.

Solution: $x^3 + 3x - 5 = x^3 + 0x^2 + 3x - 5$

\therefore given polynomial in coefficient form is $(1, 0, 3, -5)$

Ex (2) $(2, -1, 0, 5, 6)$ is the coefficient form of the polynomial. Represent it in index form.

Solution: Coefficient form of the polynomial is $(2, -1, 0, 5, 6)$

\therefore index form of the polynomial is $2x^4 - x^3 + 0x^2 + 5x + 6$ i.e. $2x^4 - x^3 + 5x + 6$

Practice set 3.1

1. State whether the given algebraic expressions are polynomials ? Justify.

- (i) $y + \frac{1}{y}$ (ii) $2 - 5\sqrt{x}$ (iii) $x^2 + 7x + 9$
 (iv) $2m^2 + 7m - 5$ (v) 10

2. Write the coefficient of m^3 in each of the given polynomial.

- (i) m^3 (ii) $\frac{-3}{2} + m - \sqrt{3}m^3$ (iii) $\frac{-2}{3}m^3 - 5m^2 + 7m - 1$

3. Write the polynomial in x using the given information.

- (i) Monomial with degree 7 (ii) Binomial with degree 35
 (iii) Trinomial with degree 8

4. Write the degree of the given polynomials.

- (i) $\sqrt{5}$ (ii) x° (iii) x^2 (iv) $\sqrt{2}m^{10} - 7$ (v) $2p - \sqrt{7}$
 (vi) $7y - y^3 + y^5$ (vii) $xyz + xy - z$ (viii) $m^3n^7 - 3m^5n + mn$

5. Classify the following polynomials as linear, quadratic and cubic polynomial.

- (i) $2x^2 + 3x + 1$ (ii) $5p$ (iii) $\sqrt{2}y - \frac{1}{2}$
 (iv) $m^3 + 7m^2 + \frac{5}{2}m - \sqrt{7}$ (v) a^2 (vi) $3r^3$

6. Write the following polynomials in standard form.

- (i) $m^3 + 3 + 5m$ (ii) $-7y + y^5 + 3y^3 - \frac{1}{2} + 2y^4 - y^2$

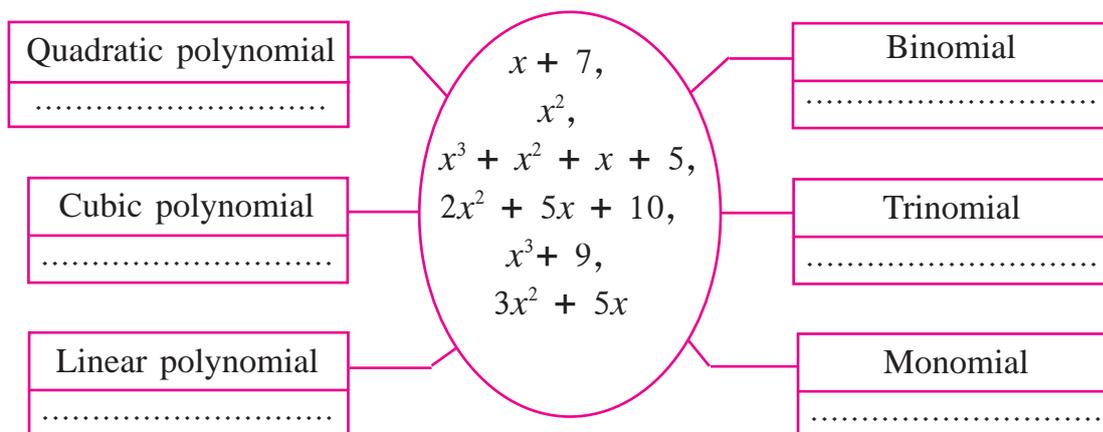
7. Write the following polynomials in coefficient form.

- (i) $x^3 - 2$ (ii) $5y$ (iii) $2m^4 - 3m^2 + 7$ (iv) $-\frac{2}{3}$

8. Write the polynomials in standard form.

- (i) (1, 2, 3) (ii) (5, 0, 0, 0, -1) (iii) (-2, 2, -2, 2)

9. Write the appropriate polynomials in the boxes.



(1) Coefficients are added or subtracted while adding or subtracting like algebraic terms, e.g. $5m^3 - 7m^3 = (5 - 7)m^3 = -2m^3$

(2) While multiplying or dividing two algebraic terms, we multiply or divide their coefficients. We also use laws of indices.

$$-4y^3 \times 2y^2z = -8y^5z ; 12a^2b \div 3ab^2 = \frac{4a}{b}$$



Let's learn.

Operations on polynomials

The methods of addition, subtraction, multiplication and division of polynomials is similar to the operation of algebraic expressions.

Ex (1) Subtract : $5a^2 - 2a$ from $7a^2 + 5a + 6$.

$$\begin{aligned} \text{Solution : } & (7a^2 + 5a + 6) - (5a^2 - 2a) \\ & = \underline{7a^2 + 5a + 6} - \underline{5a^2 + 2a} \\ & = 7a^2 - 5a^2 + 5a + 2a + 6 \\ & = 2a^2 + 7a + 6 \end{aligned}$$

Ex (2) Multiply : $-2a \times 5a^2$

$$\text{Solution : } -2a \times 5a^2 = -10a^3$$

Ex (3) Multiply : $(m^2 - 5) \times (m^3 + 2m - 2)$

$$\begin{aligned} \text{Solution : } & (m^2 - 5) \times (m^3 + 2m - 2) \\ & = m^2(m^3 + 2m - 2) - 5(m^3 + 2m - 2) \\ & = m^5 + 2m^3 - 2m^2 - 5m^3 - 10m + 10 \\ & = m^5 + 2m^3 - 5m^3 - 2m^2 - 10m + 10 \quad (\text{Like terms taken together.}) \\ & = m^5 - 3m^3 - 2m^2 - 10m + 10 \end{aligned}$$

} (Each term of second polynomials is multiplied by each term of first polynomial.)

Here the degree of the product is 5.

Ex (4) Add : $3m^2n + 5mn^2 - 7mn$ and $2m^2n - mn^2 + mn$.

$$\begin{aligned} \text{Solution : } & (3m^2n + 5mn^2 - 7mn) + (2m^2n - mn^2 + mn) \\ & = 3m^2n + 5mn^2 - 7mn + 2m^2n - mn^2 + mn \\ & = \underline{3m^2n + 2m^2n} + \underline{5mn^2 - mn^2} - \underline{7mn + mn} \quad (\text{Like terms are arranged.}) \\ & = 5m^2n + 4mn^2 - 6mn \quad (\text{Like terms are added.}) \end{aligned}$$



Let's recall.

Degree of one polynomial is 3 and the degree of other polynomials is 5. Then what is the degree of their product ?

What is the relation between degree of multiplicand and degree of a multiplier with degree of their product ?

Ex (5) Divide $(2 + 2x^2) \div (x + 2)$ and write the answer in the given form.

$$\text{Dividend} = \text{Divisor} \times \text{Quotient} + \text{Remainder}$$

Solution : Let us write the polynomial in standard form. $p(x) = 2 + 2x^2$

	$2 + 2x^2 = 2x^2 + 0x + 2$	
Method I :	$\begin{array}{r} 2x - 4 \\ x + 2 \overline{) 2x^2 + 0x + 2} \\ \underline{- 2x^2 + 4x} \\ - 4x + 2 \\ \underline{- 4x - 8} \\ + \\ \hline 10 \end{array}$	<p>Dividend = divisor \times quotient + remainder</p> <p>$2 + 2x^2 = (x + 2) \times (2x - 4) + 10$</p> <p>$q(x)$, divisor = $(x + 2)$</p> <p>$s(x)$, quotient = $2x - 4$ and</p> <p>$r(x)$, remainder = 10</p> <p>$\therefore p(x) = q(x) \times s(x) + r(x)$.</p>

Method II : Linear method of division :

Divide $(2x^2 + 2) \div (x + 2)$

To get the term $2x^2$ multiply $(x + 2)$ by $2x$ and subtract $4x$.

$$2x(x+2) - 4x = 2x^2$$

$$\therefore \text{Dividend} = 2x^2 + 2 = 2x(x+2) - 4x + 2 \quad \dots\text{(I)}$$

To get the term $-4x$ multiply $(x+2)$ by -4 and add 8.

$$-4(x+2) + 8 = -4x$$

$$\therefore (2x^2 + 2) = 2x(x+2) - 4(x+2) + 8 + 2 \quad \dots\text{from (I)}$$

$$\therefore (2x^2 + 2) = (x + 2) (2x - 4) + 10$$

Dividend = divisor \times quotient + remainder.



Remember this !

Euclid's division lemma

If $s(x)$ and $p(x)$ are two polynomials such that degree of $s(x)$ is greater than or equal to the degree of $p(x)$ and after dividing $s(x)$ by $p(x)$ the quotient is $q(x)$ then $s(x) = p(x) \times q(x) + r(x)$, where $r(x) = 0$ or degree of $r(x) < 0$.

Practice set 3.2

- (1) Use the given letters to write the answer.
 - (i) There are 'a' trees in the village Lat. If the number of trees increases every year by 'b', then how many trees will there be after 'x' years?
 - (ii) For the parade there are y students in each row and x such row are formed. Then, how many students are there for the parade in all ?
 - (iii) The tens and units place of a two digit number is m and n respectively. Write the polynomial which represents the two digit number.
- (2) Add the given polynomials.
 - (i) $x^3 - 2x^2 - 9$; $5x^3 + 2x + 9$
 - (ii) $-7m^4 + 5m^3 + \sqrt{2}$; $5m^4 - 3m^3 + 2m^2 + 3m - 6$
 - (iii) $2y^2 + 7y + 5$; $3y + 9$; $3y^2 - 4y - 3$
- (3) Subtract the second polynomial from the first.
 - (i) $x^2 - 9x + \sqrt{3}$; $-19x + \sqrt{3} + 7x^2$
 - (ii) $2ab^2 + 3a^2b - 4ab$; $3ab - 8ab^2 + 2a^2b$
- (4) Multiply the given polynomials.
 - (i) $2x$; $x^2 - 2x - 1$ (ii) $x^5 - 1$; $x^3 + 2x^2 + 2$ (iii) $2y + 1$; $y^2 - 2y^3 + 3y$
- (5) Divide first polynomial by second polynomial and write the answer in the form 'Dividend = Divisor \times Quotient + Remainder'.
 - (i) $x^3 - 64$; $x - 4$ (ii) $5x^5 + 4x^4 - 3x^3 + 2x^2 + 2$; $x^2 - x$
- (6*) Write down the information in the form of algebraic expression and simplify.

There is a rectangular farm with length $(2a^2 + 3b^2)$ metre and breadth $(a^2 + b^2)$ metre. The farmer used a square shaped plot of the farm to build a house. The side of the plot was $(a^2 - b^2)$ metre. What is the area of the remaining part of the farm ?

Activity : Read the following passage, write the appropriate amount in the boxes and discuss.

Govind, who is a dry land farmer from Shiralas has a 5 acre field. His family includes his wife, two children and his old mother. He borrowed one lakh twenty five thousand rupees from the bank for one year as agricultural loan at 10 p.c.p.a. He cultivated soyabean in x acres and cotton and tur in y acres. The expenditure he incurred was as follows :

He spent Rs. 10,000 on seeds. The expenses for fertilizers and pesticides for the soyabean crop was $2000x$ rupees and $4000x^2$ rupees were spent on wages and cultivation of land. He spent $8000y$ rupees on fertilizers and pesticides and rupees $9000y^2$ for wages and cultivation of land for the cotton and tur crops.

Let us write the total expenditure on all the crops by using variables x and y .

$$\boxed{} + \boxed{2000x} + \boxed{4000x^2} + \boxed{8000y} + \boxed{} \text{ rupees}$$

He harvested $5x^2$ quintals soyabean and sold it at Rs. 2800 per quintal. The cotton crop yield was $\frac{5}{3}y^2$ quintals which fetched Rs. 5000 per quintal. The tur crop yield was $4y$ quintals and was sold at Rs. 4000 per quintal. Let us write the total income in rupees that was obtained by selling the entire farm produce, with the help of an expression using variables x and y .

$$\boxed{} + \boxed{} + \boxed{} \text{ rupees}$$



Let's learn.

Synthetic division

We know, how to divide one polynomial by other polynomial. Now we will learn an easy method for division of polynomials when divisor is of the form $x + a$ or $x - a$.

Ex (1) Divide the polynomial $(3x^3 + 2x^2 - 1)$ by $(x + 2)$.

Solution : Let us write the dividend polynomial in the coefficient form.

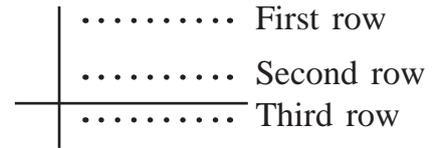
Index form of the dividend polynomial is $3x^3 + 2x^2 - 1 = 3x^3 + 2x^2 + 0x - 1$

\therefore coefficient form of the given polynomial = $(3, 2, 0, -1)$

Divisor polynomial = $x + 2$

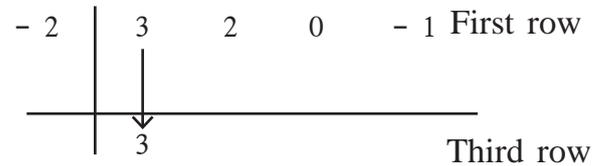
Let us use the following steps for synthetic division.

(1) Draw one horizontal and one vertical line as shown alongside.



(2) Divisor is $x + 2$. Hence take opposite number of 2 which is -2

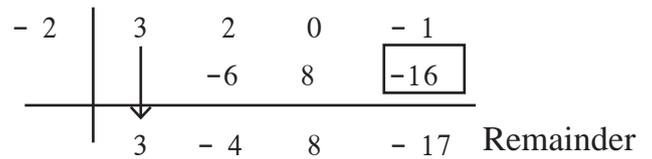
Write -2 to the left of the vertical line as shown. Write the coefficient form of the dividend polynomial in the first row.



(3) Write the first coefficient as it is in the third row.

(4) The product of 3 in the third row with divisor -2 is -6 . Write this -6 in the second row below the coefficient 2.

Addition of 2 and -6 which is -4 , is to be written in the third row.



Similarly by multiplying and adding, last addition is the remainder, which is (-17) and coefficient form of the Quotient is $(3, -4, 8)$.

$$\therefore \text{Quotient} = 3x^2 - 4x + 8 \text{ and } \text{Remainder} = -17$$

$$\therefore 3x^3 + 2x^2 - 1 = (x + 2)(3x^2 - 4x + 8) - 17$$

This method is called the **method of synthetic division**. The same division can be done by linear method of division as shown below.

$$\begin{aligned} 3x^3 + 2x^2 - 1 &= 3x^2(x + 2) - 6x^2 + 2x^2 - 1 \\ &= 3x^2(x + 2) - 4x^2 - 1 \\ &= 3x^2(x + 2) - 4x^2 - 8x + 8x - 1 \\ &= 3x^2(x + 2) - 4x(x + 2) + 8x - 1 \\ &= 3x^2(x + 2) - 4x(x + 2) + 8x + 16 - 16 - 1 \\ &= 3x^2(x + 2) - 4x(x + 2) + 8(x + 2) - 17 \end{aligned}$$

$$\therefore 3x^3 + 2x^2 - 1 = (x + 2)(3x^2 - 4x + 8) - 17$$

Ex (2) Divide $(2y^4 - 3y^3 + 5y - 4) \div (y - 1)$

Solution : Synthetic division : Dividend = $2y^4 - 3y^3 + 5y - 4 = 2y^4 - 3y^3 + 0y^2 + 5y - 4$

Divisor = $y - 1$ Opposite of -1 is 1 .

1	2	- 3	0	5	- 4	
		2	- 1	- 1	4	
	2	- 1	- 1	4	0	Remainder

Coefficient form of the quotient is $(2, -1, -1, 4)$.

\therefore Quotient = $2y^3 - y^2 - y + 4$ and Remainder = 0

Linear method : $2y^4 - 3y^3 + 5y - 4 = 2y^3(y - 1) + 2y^3 - 3y^3 + 5y - 4$

$$= 2y^3(y - 1) - y^2(y - 1) - y^2 + 5y - 4$$

$$= 2y^3(y - 1) - y^2(y - 1) - y(y - 1) + 4y - 4$$

$$= (2y^3 - y^2 - y + 4)(y - 1)$$



Remember this !

In the division by synthetic method the divisor polynomial is in the form $x + a$ or $x - a$ whose degree is 1 .

Practice set 3.3

1. Divide each of the following polynomials by synthetic division method and also by linear division method. Write the quotient and the remainder.

(i) $(2m^2 - 3m + 10) \div (m - 5)$

(ii) $(x^4 + 2x^3 + 3x^2 + 4x + 5) \div (x + 2)$

(iii) $(y^3 - 216) \div (y - 6)$

(iv) $(2x^4 + 3x^3 + 4x - 2x^2) \div (x + 3)$

(v) $(x^4 - 3x^2 - 8) \div (x + 4)$

(vi) $(y^3 - 3y^2 + 5y - 1) \div (y - 1)$



Let's learn.

Value of a polynomial

In a polynomial if variable is replaced by a number then we get the value of that polynomial. For example if we replace x by 2 in the polynomial $x + 7$ we get $2 + 7 = 9$ which is the value of that polynomial.

If $p(x)$ is a polynomial in x then the value of the polynomial for $x = a$ is written as $p(a)$.

Ex (1) Find the value of the polynomial $p(x) = 2x^2 - 3x + 5$ for $x = 2$.

Solution : Polynomial $p(x) = 2x^2 - 3x + 5$

Put $x = 2$ in the given polynomial,

$$\begin{aligned}\therefore p(2) &= 2 \times 2^2 - 3 \times 2 + 5 \\ &= 2 \times 4 - 6 + 5 \\ &= 8 - 6 + 5 \\ \therefore p(2) &= 7\end{aligned}$$

Ex (2) Find the value of $p(y) = 2y^3 - 2y + \sqrt{7}$ for $y = -2$

Solution : $p(y) = 2y^3 - 2y + \sqrt{7}$

$$\begin{aligned}\therefore p(-2) &= 2 \times (-2)^3 - 2 \times (-2) + \sqrt{7} \\ &= 2 \times (-8) - 2 \times (-2) + \sqrt{7} \\ &= -16 + 4 + \sqrt{7} \\ &= -12 + \sqrt{7}\end{aligned}$$

\therefore For $y = -2$ the value of polynomial is $-12 + \sqrt{7}$.

Ex (3) If $p(x) = 2x^2 - x^3 + x + 2$ then find $p(0)$.

Solution : $p(x) = 2x^2 - x^3 + x + 2$

$$\begin{aligned}\therefore p(0) &= 2 \times 0^2 - 0^3 + 0 + 2 \\ &= 2 \times 0 - 0 + 0 + 2 \\ &= 2\end{aligned}$$

Ex (4) If the value of the polynomial $m^2 - am + 7$ for $m = -1$ is 10, then find the value of a .

Solution : $p(m) = m^2 - am + 7$

$$\begin{aligned}\therefore p(-1) &= (-1)^2 - a \times (-1) + 7 \\ &= 1 + a + 7 \\ &= 8 + a\end{aligned}$$

But $p(-1) = 10$ (given.)

$$\begin{aligned}\therefore 8 + a &= 10 \\ \therefore a &= 10 - 8 \\ \therefore a &= 2\end{aligned}$$

Now we will find the relation between remainder and value of the polynomial as follows:

In the dividend polynomial $4x^2 - x + 2$ put $x = -1$.

$$p(x) = 4x^2 - x + 2$$

$$\begin{aligned} \therefore p(-1) &= 4 \times (-1)^2 - (-1) + 2 \\ &= 4 \times 1 + 1 + 2 \\ &= 4 + 1 + 2 \\ &= 7 \end{aligned}$$

\therefore value of the polynomial $p(x)$ for $x = -1$ is 7. (II)

From the statement (I) and (II), the remainder when $p(x) = 4x^2 - x + 2$ is divided by $(x + a)$ that is $x + 1$ and the value of the polynomial $p(x)$ for $x = -1$, that is $p(-1)$, both are equal.

Hence we get the following property.

If the polynomial $p(x)$ is divided by $(x + a)$ then the **remainder** is $p(-a)$ means it is same as the **value of the polynomial** $p(x)$ for $x = -a$

This is known as the **Remainder theorem**.

Let's prove the theorem using Euclid's division lemma.

If $p(x)$ is divided by $(x + a)$

$$p(x) = q(x) \times (x + a) + r(x) \quad [q(x) = \text{Quotient}, r(x) = \text{Remainder}]$$

If, $r(x) \neq 0$, then by rule the degree of the polynomial $r(x)$ is less than 1 means 0. Therefore $r(x)$ is a real number.

$\therefore r(-a)$ is also a real number.

$$\text{Now, } p(x) = q(x) \times (x + a) + r(x) \text{(I)}$$

By putting $x = -a$ in (I) we get,

$$\begin{aligned} p(-a) &= q(-a) \times (a - a) + r(-a) \\ &= q(-a) \times 0 + r(-a) \text{(II)} \\ \therefore p(-a) &= r(-a) \text{from (I) and (II)} \end{aligned}$$